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BASIS AND DIMENSION

Linearly Independent Set: A finite set of vectors $v_1, v_2, \dots \dots \dots . v_n$ of a vector space V over a field F is said to be linearly independent if $c_1v_1 + c_2v_2 + \cdots \dots \dots . + c_nv_n = \theta$, $c_i \in F, i = 1,2, \dots \dots n \rightarrow c_i = 0 \ \forall i$ otherwise linearly dependent.

- If the set $S = \{v_1, v_2, \dots \dots \dots . v_n\}$ of vectors of the vector space V over a field F be linearly independent then none of the vectors $v_1, v_2, \dots \dots \dots . v_n$ can be a zero vector.
- A set of vectors containing the null vector θ in a vector space $V(F)$ is linearly dependent.
- The set consisting of a single non-zero vector α in a $V(F)$ is linearly independent.
- If two vectors be linearly dependent,then one of them is a scalar multiple of the other.

Spanning Set(Linear Span): Let V be a vector space over a field F and S be any non-empty subset of V, Then the linear span of S is defined as the set of all linear combination of the elements of S and denoted by $L(S)$.

BASIS

Definition: A basis S of a vector space V over a field F is a linearly independent subset of V that spans V. This means that a subset S of V is a basis if it satisfies the two following conditions:

- The linear independence property:
  For every finite subset $\{v_1, v_2, \dots \dots \dots . v_m\}$ of S if
  $c_1v_1 + c_2v_2 + \cdots \dots \dots . + c_mv_m = \theta$ for some $c_1 = c_2 = \cdots \dots = c_m = 0$ and
- The spanning property:
  For every vector v in V , we can choose $a_1, a_2, \dots \dots ., a_n$ in F and $v_1, v_2, \dots \dots \dots . v_n$ in S such that $v = a_1v_1 + a_2v_2 + \cdots \dots \dots . + a_nv_n$
  The scalars a_i are called the coordinates of the vector v with respect to the basis S .

Examples:

1. The set

$$A = \{(1,0,0,, \dots \dots 0), (0,1,0,, \dots ..0), \dots .. (0,0,0, \dots ..1,0), (0,0,0, \dots \dots ..0,1)\}$$

is a basis of the n-dimensional vector space.This is called the standard basis .

2. The infinite set $S = \{1, x, x^2, \dots \dots , x^n, \dots ..\}$ is a basis of a vector space $P(x)$ of all polynomials over a field F.
3. The real square matrix of second order

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a linear vector space, and that a basis of it is the subset $S = \{\alpha, \beta, \gamma, \delta\}$, where $\alpha, \beta, \gamma, \delta$ are matrices

$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\delta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Properties:

- If B is a linearly independent subset of a spanning set Lsubset of V, then there is a basis S such that $B \subset S \subset L$ that means
  If $\{v_1, v_2, \dots \dots \dots . v_m\}$ be a basis of a finite dimensional vector space V over a field F then any linearly independent set of vectors in V contains at most m vectors.
- There exists atleast a basis for every finitely generated vector space.
- If there exist more than one basis of a vector space $V(F)$,all bases of $V(F)$ have the same cardinality, which is called the dimension of V.
- A generating set S is a basis of V iff it is minimal, i.e. no proper subset of S is also a generating set of V.
- A linearly independent set L is a basis iff it is maximal, i.e. it is not a proper subset of any linearly independent set.

DIMENSION

Definition: The number of vectors in a basis of a vector space V is said to be the dimension (or rank) of V and is denoted by $dim\, V$. The null space $\{\theta\}$ is said to be of dimension 0.

Examples:

1) The dimension of the vector space R^2 is 2 ,since $E = \{(1,0), (0,1)\}$ is a basis.
2) The dimension of the vector space $R_{m \times n}$ of all $m \times n$ real matrices is mn, since the set $\{E_{11}, E_{12}, \dots \dots \dots . E_{mn}\}$,where E_{ij} is an $m \times n$matrix having 1 as the $ij\ th$ element and 0 elsewhere, is a basis.
3) The dimension of the vector space P_n of all real polynomials in x of $degree < n$ together with the zero polynomial, is n. The set of polynomials $\{1, x, x2, \dots \dots xn - 1\}$ is a basis.
4) The vector space P of all real polynomials is infinite dimensional.

Worked Examples:

1. For what real values of k does the set $S = \{(k, 0,1), (1, k + 1,1), (1,1,1)\}$ form a basis of R^3.

Solution: Since dimension of R^3 is 3 and the no of element of the set S is 3 so S form a basis if S is linearly independent set.

i.e. $\begin{vmatrix} k & 0 & 1 \\ 1 & k+1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

i.e. $k\{(k + 1) - 1\} + 1\{1 - (k + 1)\} \neq 0$

i.e.$k^2 - k \neq 0$
i.e.$k(k-1) \neq 0$

Therefore for $k \neq 0,1$ the set S form a basis of R^3.

2. Let $\{\alpha, \beta, \gamma\}$ be a basis of a real vector space V and c be a non-zero real number. Prove that

$\{\alpha + c\beta, \beta, \gamma\}$ is a basis of V.

Solution: Since $\{\alpha, \beta, \gamma\}$ be a basis of V so the dimension of V is 3.

So $\{\alpha + c\beta, \beta, \gamma\}$ is a basis of V if we show that this set is linearly independent.

Now ,$c_1(\alpha + c\beta) + c_2\beta + c_3\gamma = \theta$

i.e. $c_1\alpha + (c_1c + c_2)\beta + c_3\gamma = \theta$ ……….(1)

since $\{\alpha, \beta, \gamma\}$ be a basis of V so α, β, γ are linearly independent.

Therefore, from (1) $c_1 = 0, c_1c + c_2 = 0, c_3 = 0$

i.e.$c_1 = c_2 = c_3 = 0$

so, $\{\alpha + c\beta, \beta, \gamma\}$ is a linearly independent set and therefore is a basis of V.

Replacement Theorem:

If $\{v_1, v_2, \dots \dots \dots, v_n\}$ be a basis of a vector space V over a field F and a non-zero vector β of V is expressed as $\beta = a_1v_1 + a_2v_2 + \cdots \dots \dots . + a_nv_n, a_i \in F$, then if $a_j \neq 0, \{v_1, v_2, \dots, v_{j-1}, \beta, v_{j+1} \dots \dots, v_n\}$ is a new basis of V.[That is, β can replace v_j in the basis.]

Worked Example:

Find a basis for the vector space R^3,that contains the vectors$(1,2,1)$ and $((3,6,2)$.

$R^3$is a vector space of dimension3.The standard basis for $R^3$is $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ $where$ $\epsilon_1 = (1,0,0), \epsilon_2 = (0,1,0), \epsilon_3 = (0,0,1)$.

Let$v_1 = (1,2,1), v_2 = (3,6,2)$.Then,

$$v_1 = 1\,\epsilon_1 + 2\,\epsilon_2 + 1\,\epsilon_3$$

Since the coefficients of ϵ_1 in the representation of $v_1$is non-zero, by Replacement theorem v_1 can replace ϵ_1 in the basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\{v_1, \epsilon_2, \epsilon_3\}$ can be a new basis for R^3.

Let

$$v_2 = c_1v_1 + c_2\,\epsilon_2 + c_3\,\epsilon_3$$

$$(3,6,2) = c_1(1,2,1) + c_2(0,1,0) + c_3(0,0,1)$$

So, $c_1 = 3, c_2 = 0, c_3 = -1$

Since the coefficient of $\in_3$ is non-zero, by replacement theorem v_2 can replace $\in_3$ in the basis $\{v_{1,} \in_{2,} \in_3\}$ and $\{v_{1,} \in_{2,}\ v_2\}$ can be a new basis for R^3.

Some Important Results on Basis and Dimension:

Let V be a finite dimensional vector space over a field F and $S = \{v_1, v_2, \dots\dots\dots . v_n\}$be a basis of Vi.e. the dimension of the vector space is n,then

1. Any subset of V containing more than n-vectors must be dependent.
2. Any subset of V containing less than n-vectors cannot span V.
3. Any two bases of the vector space V have the same number of elements.
4. A subset of V with n elements is a basis iff it is linearly independent.
5. A subset of V with n elements is a basis iff it is spanning set of V.
6. A linearly independent subset of this finite dimensional vector space V is either a basis or it can be extended to form a basis of V.
7. Every set of (n+1) vectors or more vectors is linearly dependent.

Dimension of a subspace: Let $V(F)$ be a vector space of finite dimension and W is a subspace of V. Then the $dimW$ is finite and $dimW \le dimV$.

Dimension of linear sum of subspace: IfW_1 and W_2 are two linear vector subspaces of a finite dimensional linear vector space V over a field F, then the dimension of their linear sum is

$$\dim(W_1 + W_2) = dimW_1 + dimW_2 - \dim(W_1 \cap W_2)$$

Dimension of direct sum: If W_1 and W_2 are two linear vector subspaces of a finite dimensional linear vector space V over a field F, then the dimension of their direct sum is

$$\dim(W_1 \boxplus W_2) = dimV = dimW_1 + dimW_2$$

Dimension of a quotient space: In a finite dimensional vector space $V(F)$ of dimension n, if W be a subspace of dimension m, then the dimension of the quotient space (V/W) is n-m.

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