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Topic for – Paper MTMA CC -1 (Vector, Calculus & Geometry)

Transformation of coordinates

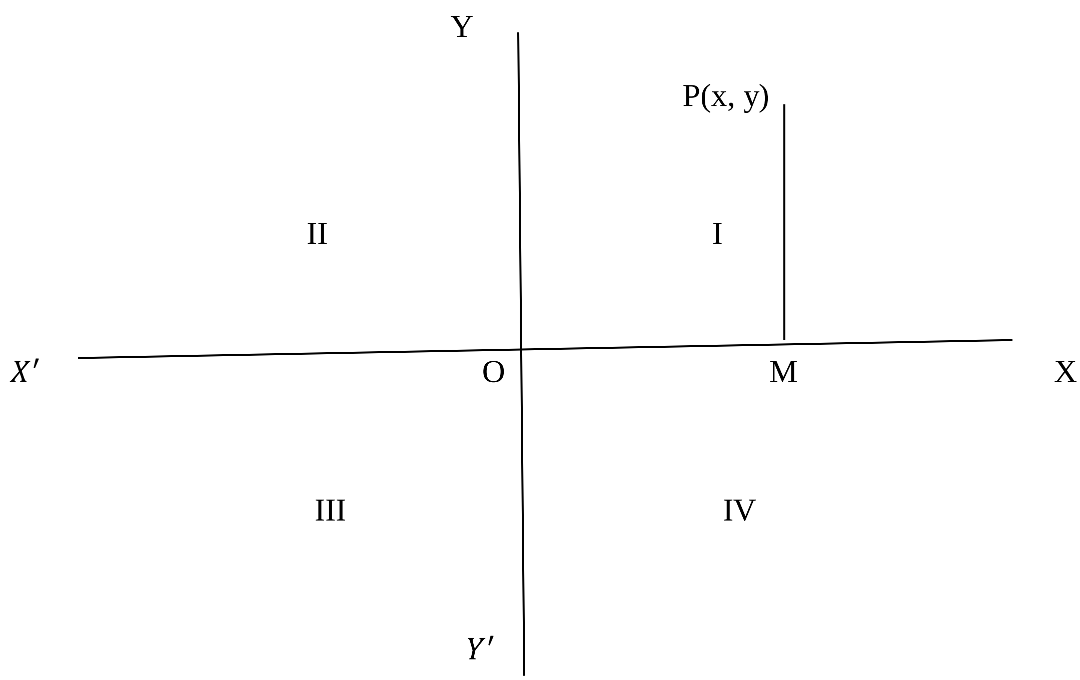
Rectangular Cartesian Coordinates

To define the position of a point space, the idea of coordinates was first invented by French Mathematician, Rene Descartes, in 1619 and in memory of the inventor co-ordinates with respect rectangular axes are sometimes called Cartesian coordinates.

First we discuss the coordinates in a plane and subsequently in space.

Let $X'OX$ and $Y'OY$ be two straight lines intersecting at right angles in the plane of the paper. The lines $X'OX$ and $Y'OY$ are called the x-axis and y-axis respectively. The two together is named as axes of coordinates. The point O is called the origin of coordinates.

From a point P in the plane of axes, PM is drawn perpendicular to OX. The position of P referred to OX. The position of P referred to OX and OY is known, if the lengths of OM and MP are known. OM and MP are called the abscissa or x-coordinate and the ordinate or y- coordinate of P respectively. These lengths with proper sign are termed as rectangular or orthogonal coordinates or simply co-ordinates of P. If x and y are lengths of OM and MP respectively then the coordinates of P are generally denoted by (x, y).



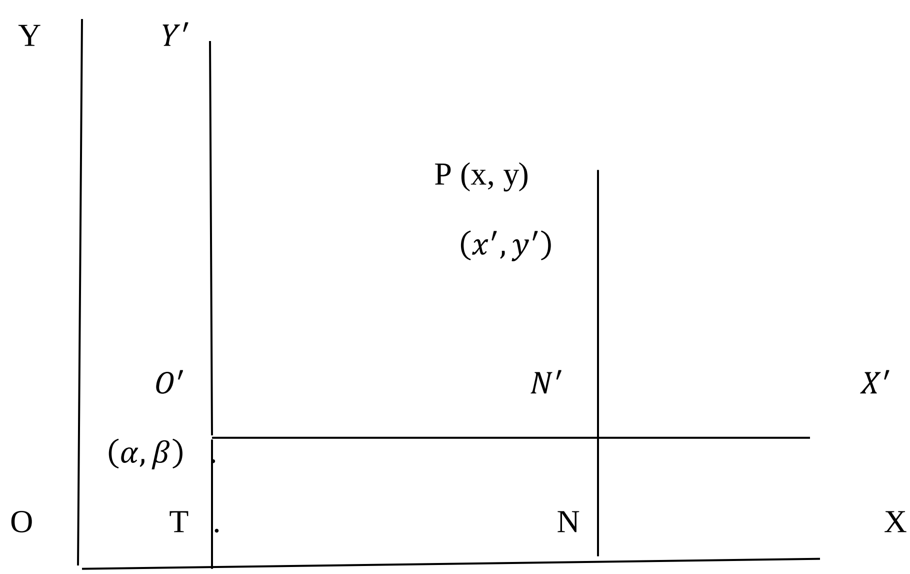
The sign of x-coordinate is positive or negative as it is measured in the direction of OX or OX'. Similarly the y-coordinate is positive or negative according as it measured in the direction of OY or OY'. If we name the regions $XOY, YOX', X'OY'$ and $Y'OX$ as quadrants I, II, III and IV respectively, then for a point in the quadrant II x-coordinate is negative and y-coordinate is positive; for that in the quadrant III both the coordinates are negative and for a point in the quadrant IV x- coordinate is positive but y-coordinate is negative. By this representation a point can be located definitely when its coordinates are given and conversely, if the point is given, its coordinates are given and conversely, if the point is given, its coordinates are definite in magnitude and sign.

Transformation of axes

The coordinates of a point depend on the position of axes. Thus the co ordinates of a point and consequently the equation of a locus will be changed with the alteration of origin without the alteration of origin without the alteration of direction of axes, or by altering the direction of axes. Either of these processes is known as transformation of coordinates.

Change of origin without change of direction of axes

Let (x, y) be the coordinates of P with respect to rectangular axes OX and OY and (x', y') be the coordinates of it with respect to a new set of axes $O'X'$ and $O'Y'$ which are parallel to the original axes OX and OY respectively.



Let (α,β) be the coordinates of the new origin O' with respect to axes OX and OY. PN is perpendicular to OX and it meets $O'X'$ at N'. $O'T$ is perpendicular to OX.

$ON = x, NP = y;\ O'N' = x',\ N'P = y';\ OT = \alpha,\ TO' = \beta.$

Now

$x = ON = ON + TN = OT + O'N' = \alpha + x',$

$y = NP = NN' + N'P = TO' + N'P = \beta + y'.$

Hence the required transformation formulae are given by $x = x' + \alpha, y = y' + \beta.$

This transformation is also known as translation or parallel displacement.

Note: In the equation of a locus referred to original system of axes (x, y) will be replaced $(x' + \alpha, y' + \beta)$ when the equation is referred to new pair of axes. Inversely (x', y') will be replaced by $(x - \alpha, y - \beta)$.

Find the equation to the curve $9x^2 + 4y^2 + 18x - 19y = 11$ referred to parallel axes through the point (-1, 2).

Solution: Here the transformation is due to shifting of the origin to the point (-1, 2).

If (x', y') be the new coordinates under translation.

$$x = x' - 1,\ y = y' + 2$$

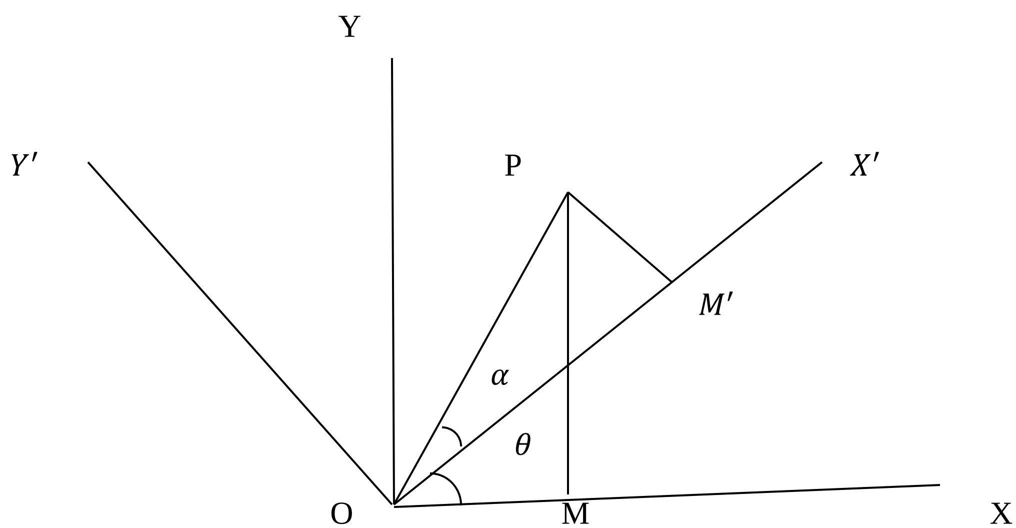
So, the transformed equation is

$9(x' - 1)^2 + 4(y' + 2) + 18(x' - 1) - 16(y' + 2) = 11$

$9x'^2 + 4y'^2 = 36$

Rotation of rectangular axes in their own plane without changing the origin

Let the original axes OX and OY be rotated through an angle θ in the anticlockwise direction. In the adjoining figure OX' and OY' are the new set of axes. Let (x, y) and (x', y') be the coordinates of the same point P referred to OX, OY and OX', OY' respectively.



PM and PM' are perpendicular to OX and OX' respectively. PO is joined. Here $\angle XOX' = \theta$. Let $\angle X'OP = \alpha$.

From the figure, $OM = x$, $MP = y; OM' = x', M'P = y'$.

Now $x = OM = OP\cos(\theta+\alpha) = OP\cos\alpha\cos\theta - OP\sin\alpha\sin\theta$

$$= OM'\cos\theta - M'P\sin\theta = x'\cos\theta - y'\sin\theta$$

$$y = MP = OP\sin(\theta+\alpha) = OP\cos\alpha\sin\theta + OP\sin\alpha\cos\theta$$

$$= OM'\sin\theta + M'P\cos\theta = x'\sin\theta + y'\cos\theta$$

Hence the change from (x, y) to (x', y') is given by

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\} \qquad \text{(i)}$$

From (i) we can easily deduce that

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= y\cos\theta - x\sin\theta \end{aligned}\right\} \qquad \text{(ii)}$$

Both of the transformations (i) and (ii) can be remembered by the scheme

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

Example. Find the equation of the line $y = \sqrt{3}x$ when the axes are rotated trough an angle $\frac{\pi}{3}$.

Solution: The transformed equation is

$$x'\sin\frac{\pi}{3} + y'\cos\frac{\pi}{3} = \sqrt{3}\left(x'\cos\frac{\pi}{3} - y'\sin\frac{\pi}{3}\right),$$

Or $\frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right),$

Or $\frac{1}{2}y' + \frac{3}{2}y' = 0,$ Or, $y' = 0$

Combination of translation and rotation

If the origin O of a set of rectangular axes (OX, OY) is shifted to $\mathrm{O}'(\alpha, \beta)$ [referred to OX and OY] without changing the direction of axes and then the axes are rotated through an angle θ in the anticlockwise direction, the total effective changes in the coordinates (x, y) of a point are given by

$$x = \alpha + x''\cos\theta - y''\sin\theta$$

and $\quad y = \beta + x'' \sin\theta + y'' \cos\theta$

(x'', y'') are the coordinates of the point referred to the final set of axes.

Transformation of coordinates when the equations of new axes are given

Let (x, y) be the coordinates of a point P referred to rectangular axes OX and OY and (x', y') be the coordinates of the same point referred to a new set of rectangular axes $\mathrm{O'X'}$ and $\mathrm{O'Y'}$ whose equations are $lx + my + n = 0$ and $mx - ly + k = 0$ with respect to OX and OY.

Perpendicular distances from P to $mx - ly + k = 0$ is

$$N'P = x' = \pm\frac{mx - ly + k}{\sqrt{l^2 + m^2}} \qquad \text{(iii)}$$

Perpendicular distance from P to $lx + my + n = 0$ is

$$M'P = y' = \pm\frac{lx + my + n}{\sqrt{l^2 + m^2}} \qquad \text{(iv)}$$

The same sign is taken in both cases according to convenience.

By (iii) and (iv) values of x and y are found out in terms of x' and y'.

Example. Find the transformed equation of the curve $(3x + 4y + 7)(4x - 3y + 5) = 50$

When the axes are $3x + 4y + 7 = 0$ and $4x - 3y + 5 = 0$.

Solution: If (x', y') be the coordinates of a point (x, y) referred to the new set of axes then

$$y' = \frac{3x + 4y + 7}{\sqrt{3^2 + 4^2}} = \frac{3x + 4y + 7}{5} \quad \text{and} \quad x' = \frac{4x - 3y + 5}{\sqrt{4^2 + 3^2}} = \frac{4x - 3y + 5}{5}.$$

The given equation can be written as

$$\frac{3x + 4y + 7}{5} \cdot \frac{4x - 3y + 5}{5} = 2$$

$$\therefore x'y' = 2$$

Hence the transformed equation is $xy = 2$.